

# **Six stages with rational Numbers**

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## **Stage 1. Free Interaction.**

We come across the implicit idea of ratio quite early in life, without realising that we are dealing with ratio. Every time we project one of our slides on to a screen in our living room, we notice that the pictures look “real”, in other words the ratios of the distances have been preserved. If we place the screen at an angle, there is distortion, and the scenes are no longer apparently “real looking”, since the ratios between the various distances have been altered. Other situations might have been barter done in childhood, such as “I’ll give you 2 of my toy soldiers for 5 of your picture books”. One child counts out 15 picture books and the other one counts out 6 toy soldiers and the exchange is completed. Or you might see a notice in a store “3 coloured pencils for 5 Dollars”, and you know that for every five dollar bill in your wallet you can get 3 coloured pencils.

Drawing is another source of such “free” experiences. A child might want to draw some houses, all the same shape, but the ones in the distance should get smaller. If the houses all have the same shape, the ratio of length and height has to be preserved as you draw the houses “further and further away”. This is done without realising that ratios are involved, such realisation will come later when precise applications are made of some of the learned relationships between ratios.

## **Stage 2. Looking for rules for ratios.**

### **Activity 1.**

You will need a pile of counters, some blue, some red, some green.

Make a pile of blue counters. Arrange them in little piles of four. If this is not possible, either add some or take some away from your pile.

Then make a pile of red counters but such that for every pile of 4 blue ones, you make a pile of 3 red ones.

Now rearrange your red counters in piles of 5. If this is not possible, go back to the beginning and start with another pile of blue counters.

Now make a pile of green counters but such that for every pile of 5 red counters you make a pile of 3 green ones.

You will now have a blue pile, a red pile and a green pile.

Now try to split your blue pile into equal piles, and also split your green pile into equal piles, but in such a way that the number of blue piles should be equal to the number of green piles.

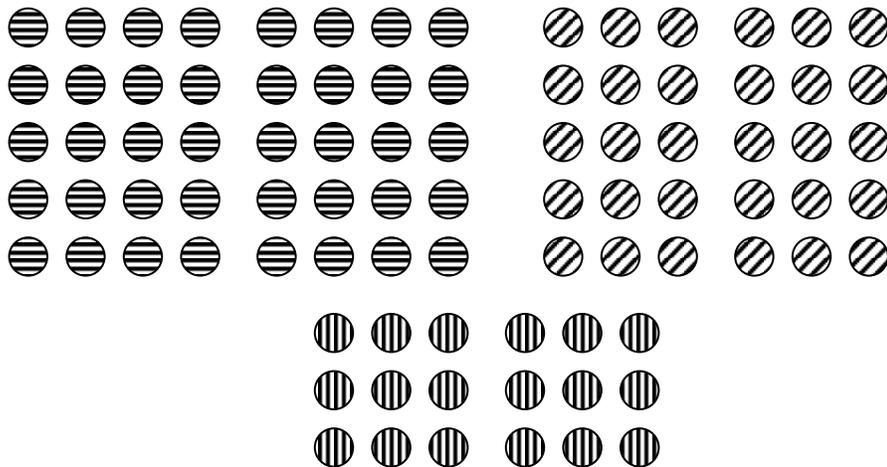
How many are there in each new little blue pile? How many are there in each little green pile?

You will now KNOW how many green counters there are in the green pile for how many blue counters in the blue pile.

These two numbers form the RATIO of the amount in the green pile to the amount in the blue pile.

If you have difficulty, look at the next part of the page.

Here is how it could be done. Let level stripes represent blue, slanting ones red and vertical ones green. Let us start with 40 blue counters:



The largest number of piles we can split the blue pile as well as the green pile into is two, if the sub-piles have to be equal in number for both coloured piles.

So the final RATIO is 9 green ones to 20 blue ones.

Now try the same sort of thing with the following RATIOS as between the red and the blue and as between the green and the red piles:

2 red for every 3 blue      3 green for every 4 red

2 red for every 5 blue      3 green for every 4 red

3 red for every 5 blue      10 green for every 9 red

each time finding out the Ratio of green to blue.

Each time you will have to look for a suitable number of counters to put in the blue pile. Whatever ratios you choose, there will always be some “good” numbers with which to start in making your pile of blue counters. Splitting the blue and the green piles into the same number of “sub-piles” will not always be possible, the “sub-piles” might have to be the actual piles themselves.

## Activity 2.

You will now need four colours. Let us use blue, red, yellow and green counters. Clear can represent yellow now.

Make a pile of blue counters. Choose the number of blue counters in such a way that you can split the whole pile up into equal little piles of 3 as well as into little piles of 4.

Now make a pile of red counters by making a little pile of 2 red counters for every little pile of 3 blue counters.

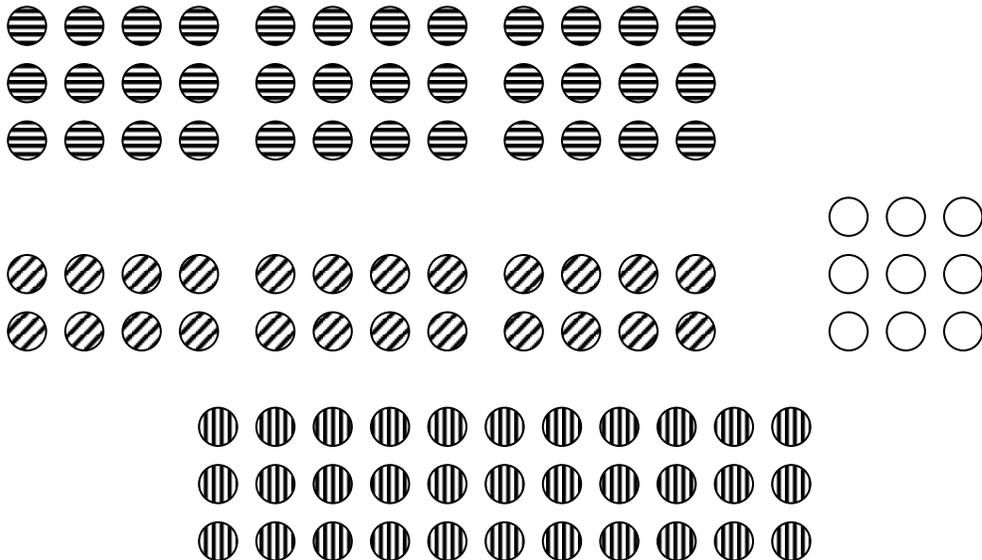
Then make a pile of yellow counters by putting one yellow counter in it for every pile of 4 blue counters.

Then make a pile of green counters in which you put just as many counters as there are in the red pile and the yellow pile together.

Then split the blue pile and the green pile into the same number of sub-piles. All the blue sub-piles must be equal in number to each other. All the green sub-piles must be equal in number to each other. And there must be exactly as many blue sub-piles as there are green sub-piles.

How many green ones are there in the green pile for how many in the blue pile ?

You could start with 36 blue counters. You will get these piles:



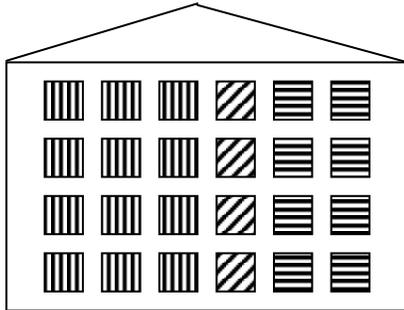
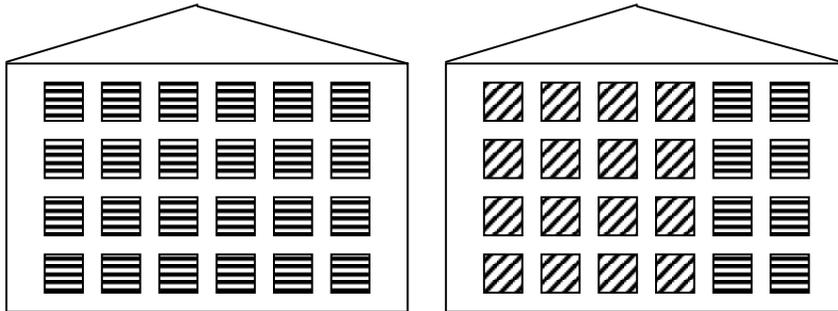
So there are 11 green ones for every 12 blue ones.

### Activity 3.

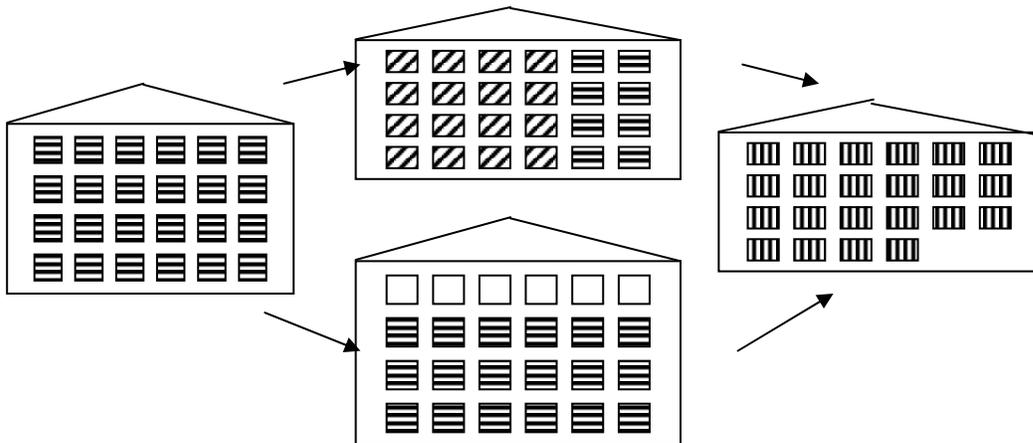
Let us draw some buildings with different kinds of windows.

Here is a house with 24 blue windows:

Then 2 out of 3 are painted red, as you can see on the right.



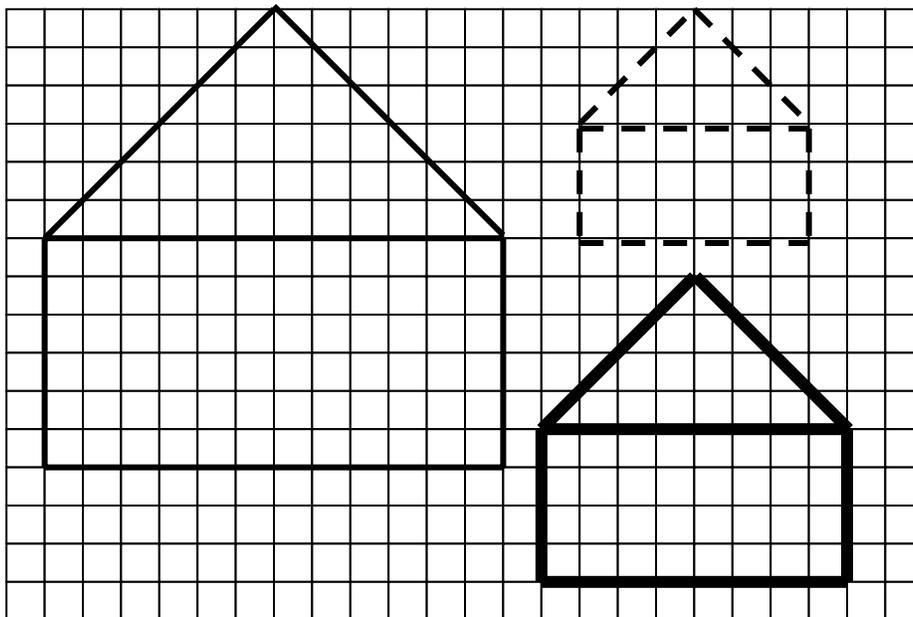
Then 3 out of 4 of the red windows are re-painted in green. Now 1 out of 2 of the windows that were blue at first are now green. You can see this by comparing the house on the left with the one above it.



Here again we start with all the windows blue. In the upper house in the middle 2 out of 3 are painted red, in the lower house in the middle 1 out of 4 is painted yellow. In the house on the right as many windows are painted green as there are red and yellow windows altogether in the middle houses. There are 11 green windows in the green house for every 12 blue windows in the blue house.

#### Activity 4.

Now let us draw some houses in a square grid. Instead of counting windows, we shall be looking at the sizes of the houses.



The big house is drawn with thin lines, the small house with dotted lines and the medium house with thick lines. The length of a side of a square in the grid is one unit length. The area of one square in the grid, is a unit area.

If you "walk along" the lines of the houses, how many units do you pass in the small house for a corresponding "walk" in the big house? This RATIO is called the linear ratio of small to big. Find the linear ratios between the houses. Do not forget that the ratio of small to big is not the same as the ratio of big to small. So there are six ratios to find.

You could also find the ratios of corresponding areas. Is there a house with 1 unit of area for every 4 units of area of another house? Try to find all six area ratios.

How are the area ratios connected to the linear ratios? If you know one as between two houses, can you immediately deduce the other? If so, how?

Draw some more houses, all the same shape, on some squared paper and try to find the linear and the area ratios between them

Draw also different objects, such as cars, flowers, trees. Make sure that any two cars you draw have the same shape, but one should be bigger than the other. You may have difficulty with trees and flowers with the area ratios, unless you draw them so that you only draw along the grid lines.

### Stage 3. Comparison of activities.

In our stage 2 work we have been doing essentially the "same thing". The purpose of this next stage is to pinpoint just exactly what this "sameness" is, in other words, we shall try to find out in what ways the activities are alike. It will be good to make up a "dictionary", so that we can "translate" what we do in one activity, into something similar that we do in another type of activity.

We can look at the work with the windows as work with the areas of the walls, as the number of windows can in a sense be considered as a measure of the area of the wall.

Here is a suggested simple "dictionary" which you can use for your "translations":

Piles of counters	Areas	Houses on the grid
Input number for first pile	Number of parts into which area is divided.	Length of a part of the first house
Output number determining the second pile	Number of these parts taken	Corresponding length in second house
Output becomes input for the next ratio	Area taken divided again into parts	Length of a part of the second house
Output number determining the third pile	Number of these parts considered	Corresponding length in the third house
Ratio between the first pile and the third pile	To get the above area, into how many parts do we divide the first area and how many of these do we take?	Ratio of corresponding lengths in the first and in the third houses.

Now you can use the above dictionary for translating one activity into one of another type. Here is an example of how you might do it.

**(A) Pile-activity.**

Make a pile of blue counters and arrange them in piles of 3.

Make a red pile with 2 red counters for every 3 blue counters in the blue pile.

Now make a green pile with 3 green counters for every 4 red ones.

You will find that there is 1 green counter in the green pile for every 2 blue counters in the blue pile.

**(B) Area-activity.**

Make a rectangle 8 cm by 6 cm, the 8 cm being the length as you look at it from left to right..

Divide your rectangle into 3 equal parts by drawing two left-to-right lines across the rectangle, one 2 cm above the "base", the other 4 cm above the "base".

Colour the upper two parts red.

Divide the red part of the rectangle into 4 equal parts by drawing two more left-to-right lines at distances 3 cm and 5 cm from the "base" of your original rectangle. Shade the upper three of these four parts black.

You will get the same part shaded, if you divide the original rectangle into two equal parts by drawing one left-to-right line 3 cm from the "base" and shading the upper one.

### (C) Houses on the grid activity.

Draw the "big house" as suggested on the grid in activity 4.

Then draw another house (call it the second one), such that 2 steps in the grid in this second house, should correspond to 3 steps in the same direction, in the "big house".

Now draw a third house in which 3 steps will correspond to 4 steps in the same direction in the second house.

A "step" can be along the grid lines or along the diagonals.

You will find that to each step in the third house, will correspond 2 equal steps in the same direction in the "big house".

A group of three could perform such tasks, each member of the group performing a task of a different type, but the corresponding actions should be carried out at the same time.

One result of such activities would be for children to realise that a ratio can be expressed in more than one way. For example

1 step in the third house for every 2 steps in the big house

could equally well be expressed as

2 steps in the third house for every 4 steps in the big house.

Similarly, it will be realised 2 for every 3 can also be expressed as 4 for every 6 or as 6 for every 9. The "same ratio" can be expressed in a great many different ways. Such ways are EQUIVALENT ways (having the same "worth") of expressing the same ratio.

## Stage 4. Ordering ratios. Representation.

How can we say whether a ratio is "bigger" or "smaller" than another ratio? This can be done by comparing the OUTPUT of each ratio, for the same INPUT. Of course we must find an INPUT that is ACCEPTABLE to both ratios. Let us look at

2 for every 3 and 3 for every 4

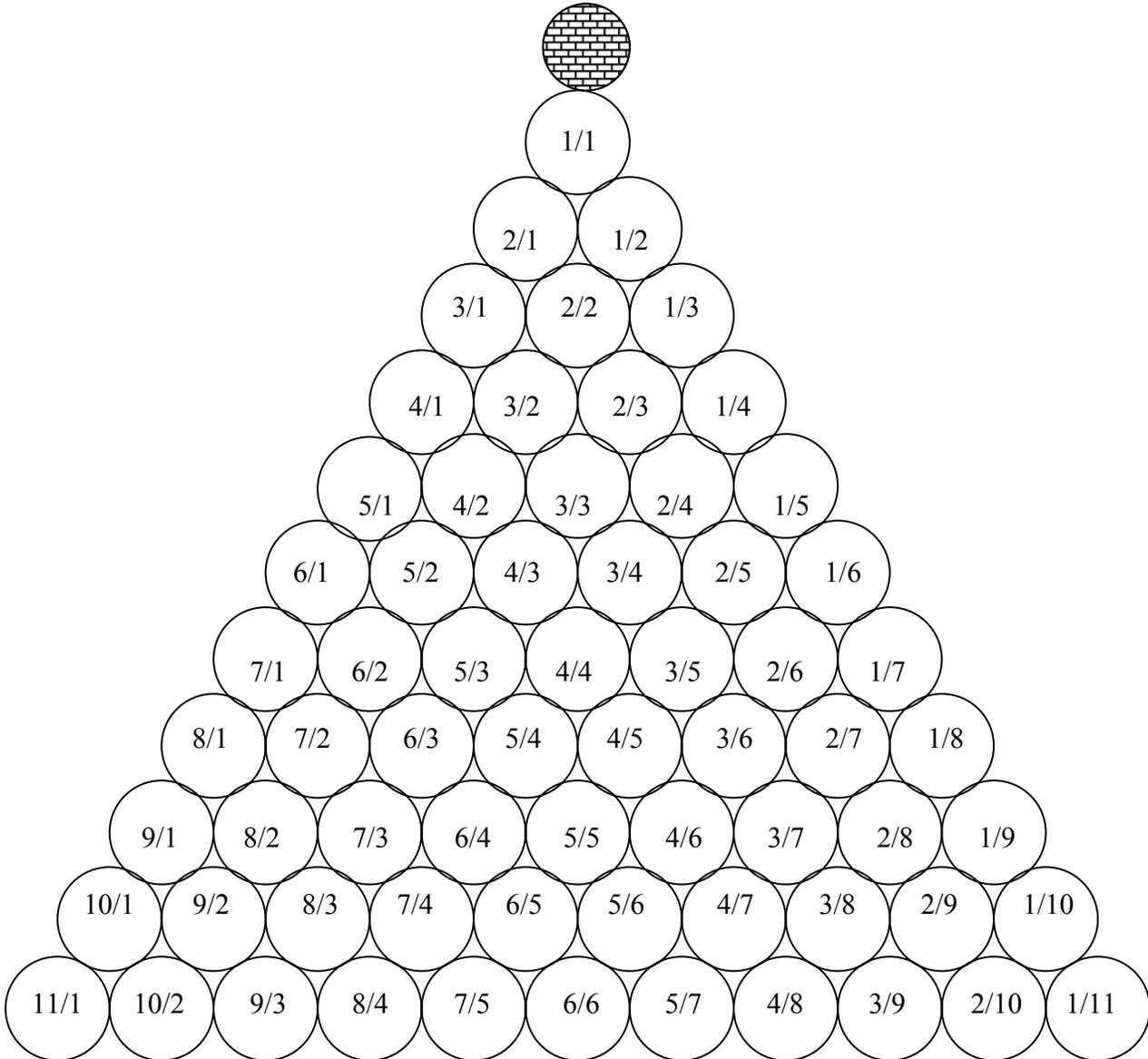
An INPUT acceptable to both of these would be, for example, 24.

Taking 2 for every 3 out of 24, we obtain the OUPUT of 16.

Taking 3 for every 4 out of 24, we obtain the OUTPUT of 18.

So "3 for every 4" is bigger than "2 for every 3"

To pinpoint how this order comes about, we can place the different ways of expressing ratios in a diagram such as the one below.



The first number is the OUTPUT and the second number is the INPUT for each ratio.

Different expressions of a ratio are known as FRACTIONS. Fractions that express the same ratio are known as EQUIVALENT FRACTIONS.

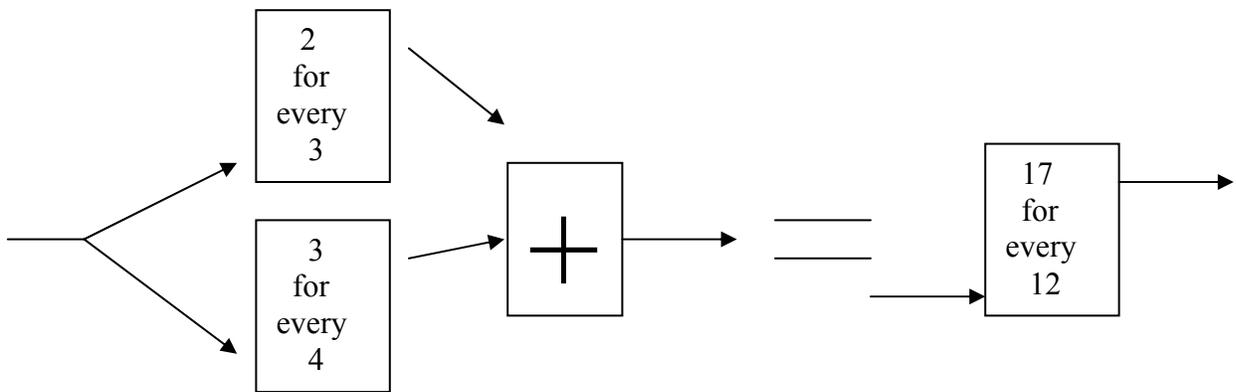
You will find equivalent fractions distributed in a kind of "fan" in the above diagram, all equivalent fractions being in line with the brick filled circle at the top.

For any two amounts A and B, A for every B and B for every A will be found symmetrically placed, being each other's image about the line down the middle of the fan.

A suitable graphical representation of obtaining a ratio out of any two given ratios could be the following:



This is what happens when the OUTPUT of the first ratio is fed in as INPUT for the second ratio. The two together are seen to perform the same "work" as the ratio on the right. This will lead to multiplication.



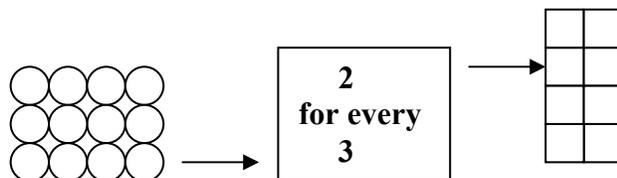
The above would be a good representation when we wish to add the OUTPUTS of two different ratios.

This procedure will lead to the addition of ratios

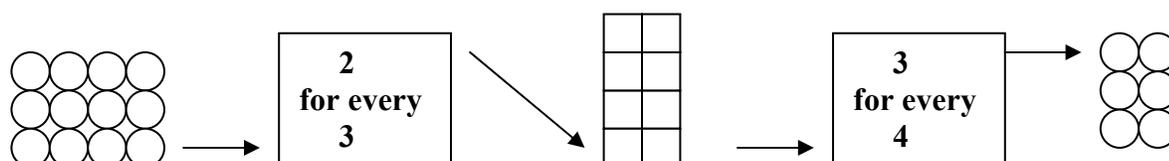
The arrow leading into a ratio represents an INPUT NUMBER, an arrow leading out of a ratio represents an OUTPUT number, resulting from what the ratio "does" to the INPUT NUMBER.

## Stage 5. Introduction of a symbol system

At this point it will be useful to introduce the conventional symbolic notation, but still with a certain amount of care. For example



could now be written as  $12 \times \frac{2}{3} = 8$



could now be written as  $12 \times \frac{2}{3} \times \frac{3}{4} = 6$

Eventually we can write the ratios as "multipliers", without writing in the actual input numbers and output numbers. So we can write things like this:

$$(x \frac{2}{3}) (x \frac{3}{4}) = (x \frac{1}{2})$$

the equal sign meaning "equivalent", namely "doing the same thing as"

The above means that the learner is now able to join two "operators" and find an equivalent "operator" to the ones that have been joined. This happens when the learner is able to pass from the pre-operational to the operational stage of the study of ratios.

At this stage the operator can become, psychologically speaking, a state, and that is how an operator can operate on it. In the pre-operational stage the learner can operate on states, but not on operators. If you are pre-operational as far as ratios are concerned, you can check whether in pile B there are 2 objects for any object in another pile A, and then whether there are 3 objects in a pile C for every object in pile B. But if you are asked how many there are in pile C for every one in pile A, you will not be able to answer without resorting the piles. But if you are operational, you can reason that 3 times 2 times a number must be 6 times that number. Your pre-operational brother will no doubt know that 2 times 3 is 6, but not that 2 times 3 times a number is 6 times that number.

In the operational stage you can regard  $(\frac{2}{3})$  as a "state of two thirds", obtained from some "generic" input that you do not have to know, known as "one unit".

So then it will make sense to write

$$(2/3) \times (3/4) = (1/2)$$

where  $(2/3)$  is a "fractional state",  $\times (3/4)$  is an operator operating on this state, and  $(1/2)$  is the resulting output state.

The same remarks apply to the adding and subtracting ratios. A pre-operational learner can feed the same input into two different ratios, and then add the outputs. He or she can then find another ratio which yields the final output number from the initial one fed into the ratios. But "adding" two ratios, without thinking of a definite input number for them, is beyond him or her. The only way in such a case is the learning of a mechanical procedure, without any idea of what is really going on. Such learning is clearly useless for any applications, unless such applications have also been mechanically learned.

## Stage 6. Formalisation.

The student of ratios will have to become familiar with a certain number of properties of "fractions". before being able to pass on to putting them into a formal system. I shall try to show how this can be done in the case of multiplication, which seems to me an easier concept than addition in the case of ratios, since ratios are really "multiplicative creatures".

Here are a few necessary insights that will need to be experienced:

Let  $p$  and  $q$  denote positive integers.

- (a)  $(p \text{ for } q) = (p \text{ for } 1) \times (1 \text{ for } q)$
- (b)  $(p \text{ for } 1) \times (q \text{ for } 1) = (p \times q \text{ for } 1)$
- (c)  $(1 \text{ for } p) \times (1 \text{ for } q) = (1 \text{ for } p \times q)$
- (d)  $(p \text{ for } q) \times (q \text{ for } p) = (1 \text{ for } 1)$  (rule for inverses)
- (e)  $(\text{Ratio } n) \times (\text{Ratio } m) = (\text{Ratio } m) \times (\text{Ratio } n)$  [Commutativity]
- (f)  $(\text{Ratio } n) \times (1 \text{ for } 1) = (\text{Ratio } n)$  (rule for multiplicative neutral)
- (g) For any three ratios  $f, g$  and  $h$   $(f \times g) \times h = f \times (g \times h)$  [Associativity]
- (h) For any three ratios  $f, g$  and  $h$  if  $f = g$  and  $g = h$  then  $f = h$  [Transitivity]

The above can be used in any deductions. Here is an example:

$$(p \times n \text{ for } q \times n) = [(p \times n \text{ for } 1) \times (1 \text{ for } (q \times n))] \text{ by using (a)}$$

$$[(p \times n \text{ for } 1) \times (1 \text{ for } (q \times n))] = (p \text{ for } 1) \times (n \text{ for } 1) \times (1 \text{ for } q) \times (1 \text{ for } n) \text{ by using (b) and (c)}$$

$$(p \text{ for } 1) \times (n \text{ for } 1) \times (1 \text{ for } q) \times (1 \text{ for } n) = (p \text{ for } 1) \times (n \text{ for } 1) \times (1 \text{ for } n) \times (1 \text{ for } q) \text{ using (e) and (g)}$$

$$(p \text{ for } 1) \times (n \text{ for } 1) \times (1 \text{ for } n) \times (1 \text{ for } q) = (p \text{ for } 1) \times (1 \text{ for } 1) \times (1 \text{ for } q) \text{ using (d)}$$

$$(p \text{ for } 1) \times (1 \text{ for } 1) \times (1 \text{ for } q) = (p \text{ for } 1) \times (1 \text{ for } q) \text{ using (f)}$$

$$(p \text{ for } 1) \times (1 \text{ for } q) = (p \text{ for } q) \text{ using (a)}$$

$$(p \times n \text{ for } q \times n) = (p \text{ for } q) \text{ by (h)}$$

If we look at the "insights" (a) to (h) as axioms, it is clear that "it follows" that we can "cancel" a common factor from the numerator and the denominator of any fractional expression of a ratio and obtain an equivalent fraction, namely one that is "equal" to the fractional expression with the common factors.

I will leave it as an amusing exercise for the reader to extend the system to "negative fractions" as well as to additions and subtractions.