Bruce and Alice learn some geometry.

In Eightland of Ruritania, children did not only learn mathematical games so as to teach them about **functions**, they also learned **logical games** so as to teach them to think better, they also learned **algebra**, but not with paper and pencil, but with trays, cups and small pebbles. They also had a go at **Geometry,** but again not in the way in which most children learn geometry. Here is Bruce, who wants to explain the first problem:

"Let me start with the first problem Alice and I were given to solve" , said Bruce.

"The following thirteen people in a party of players want to play Fourland tennis. They do not mind with whom or against whom they play, but they want to be in just enough teams of four so as they will have played in one and the same team with every other member of the party, but only once. These are the names of the players:

Adam, Ann, Betsy, Bill, Bruno, Carol, Charlie, George, John, Karen, Laurie, Mary and Michael.

Michael was organizing the games so made the following additional requirements:

Ann, Betsy and Carol do not play well together in a team, so Michael says they should never be all three together in the same foursome. The same is true of Adam, Bill and Charlie, they should never be together all three in the same team.

But Michael insists that he wants to play in a team in which Ann and Adam are included, also in another team in which Betsy and Bill are included, and finally also in a team in which Carol and Charlie are included."

"Alice and I and the other children in our class had to make up the teams, so that everyone of the thirteen players teamed up with each of the other players but only once. The teacher reminded us not to forget to satisfy Michael's requirements as well! "

"When our teams were made up (there were to be thirteen teams), we had to work out who the following players were", concluded Bruce and then wrote the following on the board:

(i) Which is the player who plays in a team with Ann and Betsy, as well as in a team with Adam and Bill?
(ii) Which is the player who plays in a team with Carol and Betsy, as well as in a team with Charlie and Bill?
(iii) Which is the player who plays in a team with Carol and Ann, as well as in a team with Adam and Charlie.

Then the teacher told them to decide whether these three players formed part of one and the same team? And if so, who was the fourth player in that team? Then they had an appoint an umpire to be responsible for each game in such a way that if any three games had a player in common, the three respective umpires should play in one and the same team. An umpire could or need not belong to the team that was playing.

If you read on, you can see how Bruce and Alice solved the problem.

Here are the thirteen teams that the children came up with. They appeared to solve the problem set by the teacher. The umpires are placed under each team, in parentheses:

Michael, Adam, Ann, Bruno (Adam)	Michael, Bill, Betsy, George, (Laurie)	Michael, Charlie, Carol, John, (Carol)
	Michael, Karen, Laurie, Mary (George)	
Adam, George, Carol, Laurie, (Michael)	Adam, Charlie, Betsy, Mary. (Bruno)	Adam, Bill, Karen, John (Ann)
Ann, John, Betsy, Laurie, (Bill)	Ann, Bill, Carol, Mary, (John)	Ann, Charlie, Karen, George (Karen)
Bruno, Betsy, Carol, Karen (Charlie)	Bruno, Bill, Charlie, Laurie, (Betsy)	Bruno, George, John, Mary (Mary)

The persons asked for under (i), (ii) and (iii) are John, Bruno and Mary.
They are in the last team listed, and the fourth player in their team is George.
The children also managed to fix things so that the three umpires of any three teams with a common player, also played in one and the same team.

The teacher was very pleased with this solution and said this to the class:
"You have now made yourself familiar with the above problems, so you are ready to begin to understand a theorem called Desargue's theorem. Here it is:" and he wrote this on the board:

Consider a triangle A B C and another triangle A' B' C', both in the same plane, but such that the lines A A', B B' and C C' meet in a point M, then the following three points:

intersection of AB and A'B', intersection of BC and B'C', intersection of AC and A'C' all belong to the same line.

At this point Bruce was perplexed. He asked the teacher the following:
"How are these triangles ABC and A'B'C' to do with playing a tennis tournament?"

"In the tennis problem", answered the teacher "the “points” are the players, and the “lines” are the teams. The one player common to two teams is the point in which the corresponding lines intersect. The team in which any two chosen players play, corresponds to the line on which the two corresponding points lie. Or, actually, you could call the teams the “points” and the players the “lines”, it would not make any difference. This is what mathematicians call DUALITY. Dual means two ways. There are two ways of looking at it"

Bruce and Alice found it a little hard to think of players as points and of the teams as lines. When Alice said as much to the teacher, he replied:

"Do you think a smudge on the board is a point, and a long line of chalk across the board a line?"

"Are they not?, questioned Unta.

"You have probably learned in your country", said the teacher, turning to Bruce, " that you can only draw one line through any two given points. But look at these two big chalk smudges", said the teacher making two big chalk blots on the board. "I can draw a number of lines through those two smudges with my ruler, if I use a sharp enough piece of chalk!"

"That is right", replied Bruce, "but if the smudges were small enough, could you then perhaps only draw one line through them?"

"However small the smudges", replied the teacher, " I can always sharpen my chalk so as to be sharp enough to draw more than one line through them! But in the tennis game, any two players (or points) belong to only one team (line). So the players and the teams are better for illustrating points and lines than smudges and chalk lines!"

"I think I am beginning to get the point", said Bruce, and they all laughed, as they thought Bruce was making a play of words on the word point.

"Are there many solutions to the problem?", asked Ata.

"You are right to ask, Ata, the solution we have here is only one of many. But let us go on to another problem, using the same thirteen players. By now Adam, Bill and Charles have made it up, and so have Anne, Betsy and Carol. Now they want to play together in one team. Michael still wants to be in one and the same team with Adam and Ann, he also wants to be in the team in which Bill and Betsy play, as well as in the team in which Carol and Charles play. Make up the teams and find out who the following players are:

(i) the player who is in the Adam-Betsy and in the Ann-Bill team,
(ii) the player who is in the Bill-Carol and in the Betsy-Charles team,
(iii) the player who is in the Adam-Carol and in the Ann-Charles team.

Are these three players playing in one and the same team?"

"Do you want us to find umpires for this problem as well?", asked Alo, Unta's brother.

"Yes, allot an umpire to each team as before", replied the teacher, "but again so that any three teams with a common player should have three umpires that play in the same team."

"Let us now put them in teams" suggested one of the children, "we can choose the umpires as before. Since the teams will be different, so will be the umpires, although everyone gets a chance to be umpire as there are thirteen players and there are bound to be thirteen teams"

After a lot of work and discussion this is what the class came up with, as a result of trying to work on the problem together:

George, Adam, Bill, Charlie (Bill) George, Ann, Betsy, Carol, (Bruno) George, Mary, Laurie, John (Mary)

George, Karen, Bruno, Michael, (Ann) Adam, Betsy, Mary, Karen, (Laurie) Adam, Carol, Laurie, Bruno (Betsy)

Adam, Ann, John, Michael, (Michael) Bill, Ann, Mary, Bruno, (George) Bill, Betsy, Laurie, Michael, (Adam)

Bill, Karen, Carol, John, (Charlie) Charlie, Betsy, Bruno, John, (Carol) Charlie, Carol, Mary, Michael, (John)

Charlie. Ann, Laurie, Karen (Karen)

"Now let us find out who the players are that we have been asked about", said Ata, "the first one is the one who plays in the Adam-Betsy team as well as in the Ann-Bill team. Let's have a look, I see, it's Mary, she plays in both teams!"

"Now we have to find out who the player is who plays in the Bill-Carol team as well as in the Betsy-Charlie team, added Unta, "these are the first two in the fourth line, and John plays in both these. The third person we want is the one who plays in the Adam-Carol team as well as in the Ann-Charlie team. These are: the last in the second row and the last team. Laurie plays in each of these."

So we need to see whether Mary, John and Laurie do or do not play in a team together", suggested Alice., "and they do! They play in the third team of our first row!. And George is the fourth player in that team."

"And George is the player that makes up the fourth for both the previously quarrelsome players!", added Bruce. "You can see that by looking at the first two teams of our first row."

"Let us just have a spot check on the umpires", said Alo, "For example George is common to our first four teams. These teams are umpired by Bill, Bruno, Mary and Ann.. These four do in fact form a team, in fact they are umpired by George! Wonders will never cease!"

"Very good!", said the teacher. "Now that you have worked out this problem for yourselves, you can begin to understand how to construct a CROSS-AXIS in the kind of geometry that Bruce and Alice are probably used to! This is how you do it:.

Imagine three points A, B and C on the same line and three points A', B' and C' on another line. The two lines have the point G in common. If the lines A A', B B' and C C' all intersect in the same point M, then the following intersections

A B' with B A', B C' with C B', A C' with C A'

are all points on the same line. This line is called the cross-axis of the two ranges of points A, B, C and A', B', C'"

"Are there any other ways of thinking of points and lines besides players and teams?", asked Ata, after all the class had gone through the construction just described several times.

"Yes, of course", said the teacher" Let me think of another way!"

"Do you just invent these things out of your head?", asked Alice.

"Yes, I think I have just invented another way!", replied the teacher. Here it is. Instead of thirteen players we could construct thirteen elements out of three letters A, B and C and a "slash" separating the letters from each other. The order in which the letters are placed is not important, but I will try to keep to an alphabetical order, except in one case. Here are the elements of the new game":

And the teacher wrote this on the chalkboard

A, B, C, A/B, B/C, A/C, AB, BC, AC, ABC, A/BC, AB/C, **B/AC**

"There are four reasons for elements to make up a set.", added the teacher. Then he wrote the reasons on the board like this:

(i) one of the letters is missing,
(ii) two of the three letters are always un-separated by a slash or neither letter is present
(iii) two of the three letters are always separated by a slash or neither letter is present,
(iv) there is one three-letter element and three two-letter elements in the set, but any two letters that are separated in the three-letter element, are not separated as they occur in the two-letter element; also any two letters that are not separated in the three-letter element, are separated as they occur in the two-letter element.

"Here are the sets of elements that are allowed to be made according to the above rules", continued the teacher, writing down these thirteen sets:

B, C, BC, B/C	A, C, AC, A/C,	A, B, AB, A/B,
BC, ABC, A/BC, A	AC, ABC, B/AC, B,	AB, ABC, AB/C, C
B/C, AB/C, B/AC, A,	A/C, A/BC, A/BC, B,	A/B, A/BC, B/AC, C
AB/C, A/B, AC, BC,	A/BC, AB, AC, B/C,	B/AC, AB, BC, A/C
	ABC, A/B, B/C, A/C	

You might like to check the following interesting facts about these thirteen sets of elements:

(i) Any two sets have one and only one element in common
(ii) Any two given elements belong to just one set,
(iii) Every set has four elements,
(iv) Every element belongs to four different sets.

"Let us now try to find a marker for each set", continued the teacher" but in such a way that any three sets having a common element should have three markers belonging to the same set. This is much easier to do in this "embodiment" of the problem than in the case of choosing umpires for our tennis matches."

"To each set in which a letter is missing, let the marker be the missing letter," he said.
"To each set defined by a pair of "undivided" letters, let us choose as marker the same pair of letters, but with a slash between them.", he said, continuing to suggest ways of finding marking elements to each kind of set.
"To each set defined by a pair of "divided" letters, let us choose as marker the same pair of letters, but without a slash between them.", he continued.
"To each set with one three-letter element and three two-letter elements, let us use the three-letter element which is a member of that same set as a marker for that set", he concluded.
"Does this solve the umpire problem?", asked Bruce.
"You can check on that for your homework!", replied the teacher

" Look at these sets [A, C, AC, A/C], B [A, B, AB, A/B], C [BC, ABC, A/BC, A] , B/C
[B/C, AB/C, B/AC, A], BC , they all have an A in each. Then look at their markers!
The marker for [B, C, B/C, BC] is A", suggested Bruce "So we are home!"

Alice put her hand up. The teacher called on her to speak..

"I noticed that you always picked a marker for every set which was in a sense “opposite” to the elements of the set." said Alice. "If a letter was missing, then you picked this same missing letter. For “undivided” sets you chose a “divided” element. For “divided” sets you seemed to want an “undivided” element . Even in the last cases, the two-letter elements are really in a sense “opposite” to how the same two letters are placed in the three-letter element of the set."

"I am glad you noticed that", said the teacher to Alice, "there is a reason for it!"

"Did you really pull all that out of your head just like that?", asked Unta "Please excuse me, but I find it hard to believe!"

"I think it is time I gave the game away", replied the teacher "For those who have studied mathematics, what follows will be no surprise. But to explain to you children where all these problems come from, I should like you to get hold of a cube, or you could make one out of cardboard. Have a black and a red pencil ready. Draw the letter A with a black pencil on one face, then draw the letter A with the red pencil on the opposite face. Draw a black B on another face and a red B on the face opposite to this one. Then draw a black C and a red C on the remaining two opposite faces."

The children got busy making their cubes, each group making a cube out of cardboard. They quickly wrote the letters A, B and C on the faces, as suggested by the teacher.

"We had better invent a language so we can talk about the cube", continued the teacher. The line joining the centres of the A-faces is the A-axis. The line joining the centres of the B-faces is the B-axis and the line joining the centres of the C-faces is the C-axis."

The teacher made sure the children could see where their axes were, on the cubes they had made. Then he went on to say:

"There are twelve edges on the cube, therefore there are six pairs of opposite edges. The midpoint of each edge can be joined to the midpoint of the opposite edge. These give us six more axes. We can use the letters on the faces touching an edge as the names of these axes. There will be a slash between the letters if the letters on these faces are of different color, and there will be no slash if the letters on these faces are of the same color. So we shall have the six axes: AB, A/B, BC, B/C, AC, A/C by doing this. Make sure you all know where your axes are and that you can call each one by its name!", concluded the teacher, who was getting out of breath with all this explaining! Then he went on:

"The four axes joining the opposite vertices will use all three letters. There will be one such axis where all the three faces that come together have the same color. This is the ABC-axis. The other axes will have three faces around them, two of them having the same color and the third one a different color. The letter which has the different color then will be separated by a slash from the other two letters. So we shall have the axes: AB/C, A/BC, B/AC."

"So each of the thirteen axes of rotation of the cube will have acquired a “name”. And these names are the thirteen elements of our second game”

The children spent the rest of the lesson learning the names of all the thirteen axes. So this is where the thirteen names came from with the letters A, B and C and a slash forming the names! The teacher obviously knew about the cube and "cribbed" the names from the axes of the cube!

After school Bruce, Alice, Alo, Ata and Unta had another good look at their cube. They looked at the various sets among the thirteen sets and tried to locate all the axes that belonged to the same set. They found that most of the time, though not always, axes belonging to the same set lay in one plane! They found that this happened with the sets coming from rules (i), (ii) and (iii). They also found that the marker element given to a set was at right angles to this plane. In fact the way they discovered this was by twiddling the cube round, holding it at each end of a "marker axis", they found that the axes of the set whose marker they were holding, would come up gradually as they turned the cube around! In the case of the sets in which there is one three-letter element and there are three two-letter elements, the axes corresponding to the two-letter elements all lie in the same plane, which is perpendicular to the axis corresponding to the three-letter element. These are the only four sets or teams of tennis players, in which the “umpire” belongs to the set or tennis team!

They also found that the plane determined by the axes corresponding to the two-letter elements, intersect the surface of the cube in the shape of a regular hexagon. The other planes, in which the axes of the remaining sets were, they found intersected the cube in a rectangle (which was a square in the case of a set with one letter missing).

The following questions were set as optional homework:

In the two problems, namely Desargue’s theorem and the cross axis, if the three “points” in question at the end are “in line”, does it follow that the “joins” of the corresponding points all meet in a point?

Can you “prove” Desargue’s theorem by using only “incidence properties” (namely about joins and intersections)?

What happens if you “cheat” and introduce a third dimension, so that two planes meet in a line, three planes that do not meet in a line meet in a point, three points determine a plane, unless they are in line etc. ? Have the triangle A B C in one plane, the triangle A’ B’ C’ in another plane, and the point in which A A’, B B’ and C C’ meet not in either plane. Then AB and A’B’, as well as AC and A’C’ as well as BC and B’C’ must meet in the line of intersection of the two planes! Can you use this fact to “prove” Desargue’s theorem for two triangles in the same plane?

If you find the “umpires” of four tennis teams that have a common player, and locate the single team in which these “umpires” play, can you tell right away who the “umpire” of that single team is?